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On possible attenuation of space singularity in modified general relativity with quantum-deformed conformal metric

A. Tawfik [*,†,‡] and Saleh O. Allehabi [*]

[*]*Department of Physics, Islamic University of Madinah,*
Madinah 42351, Saudi Arabia

[†]*Ahram Canadian University (ACU), Faculty of Engineering, 12556 Giza, Egypt*
[‡]*tawfik@itp.uni-frankfurt.de; 400778@iu.edu.sa; atawfik@bnl.gov*

A. A. Alshehri

Department of Science and Technology, University College at Nairiyah,
University of Hafr Al Batin (UHB), Nairiyah 31981, Saudi Arabia

D. Mukherjee

Indian Institute of Science Education and Research (IISER) Berhampur,
Vigyanpuri, Laudigam, Konisi, Berhampur 760003, Odisha, India

M. Nasar

Department of Physics, Faculty of Science, Benha University, 12511 Benha, Egypt

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With a geometric quantization ansatz that combines phase-space Finsler/Hamilton geometry and the Relativistic Generalized Uncertainty Principle (RGUP), the metric tensor in general relativity (GR) is approximately conformally reformulated. Thereby, different quantum-mechanical operators can be properly reconciled with the principles of GR. The phase-space Hessian associated with the squared Finsler/Hamilton structure $F(x_0, p_0)$ results in a phase-space metric tensor that can be transformed into a four-dimensional metric tensor which is approximately conformally revisited. The conformal coefficient $C(x_0, p_0, \dot{p}_0)$ depends on space–time, momentum space and the maximal proper force. By means of analytical and numerical timelike geodesic congruence associated with the Schwarzschild metric, the spherically symmetric vacuum solution of the Einstein field equations, it was concluded that the sign and magnitude of $C(x_0, p_0, \dot{p}_0)$ play a crucial role in governing the Raychaudhuri expansion: specifically, $C(x_0, p_0, \dot{p}_0) > 1$ tends to reduce focusing and thereby reduce singular behavior, while $C(x_0, p_0, \dot{p}_0) < 1$ tends to increase focusing and thereby enhance singular behavior. Therefore, the space singularity can be partially controlled or entirely regulated depending on the

[‡]Corresponding author.

approximate values assigned to $C(x_0, p_0, \dot{p}_0)$. We investigate the assumptions used to treat $F^2(x_0, p_0)$ as an effective Hamiltonian and to truncate the quantized metric tensor, as well as their relevance to the maximal proper force. Although there are considerable limitations to the proposed formulation due to various approximations, such as the estimation of quantum operators, we conclude that rigorous operator approaches open up numerous research possibilities.

Keywords: Riemann–Finsler–Hamilton discretized geometry; quantum and noncommutative differential geometry; quantum-deformed fundamental metric tensor; spherically symmetric vacuum solution of modified general relativity.

1. Introduction

There is a general consensus that the essential notion of gravitation, understood as the curvature of space–time, significantly influences our present-day interpretation of the cosmos.[1] In general relativity (GR), the local parts of four-dimensional space–time are approximated as Minkowski spaces, whose spatial sections do not have to be Euclidean. Governed by a nonvanishing cosmic substance, which can be represented by a finite stress-energy tensor, the Einstein field equations (EFE) describe the expected geometric deformations that occur on large scales. Despite the strong alignment of GR with a wide range of experimental and observational evidence, the essential geometric postulates of GR and its theoretical consistency are not always assured, particularly at relativistic energies and/or at quantum (low) scales.[2] Relativistic energy essentially indicates a high energy scale where space–time is no longer smooth and certain. An energy scale that facilitates a quantum description need not necessarily be related to the Planck energy scale, as gravity-deformed quantum mechanics (QM) by means of the generalized uncertainty principle (GUP), for instance, remarkably revisits the Planck scale. Deep-rooted mathematical problems that may have their origins in the nature of geometric structures include space and initial singularities, as well as the inherent differences from QM, the other fundamental theory of nature.

A theoretical ansatz was proposed for quantum-mechanically inspired revisions to GR by: (i) expanding classical differential geometry to quantum differential geometry and (ii) introducing a new theory on the kinematics of quantum particles within a phase-space structured manifold.[3–8] This new ansatz proposes an augmented (quantum-mechanically inspired) modification of the fundamental metric tensor motivated by the Relativistic Generalized Uncertainty Principle (RGUP)[9–11] and a profound extension of Riemann to Finsler/Hamilton geometry. It was emphasized that the framework developed here is a geometric quantization proposal (an augmentation based on the metric structure of GR), not a full theory of quantum gravity with dynamical gravitons.[12–14] The main motivation is to provide a viable resolution to the persistent problem of the fundamental distinction between the principles of GR and QM. The results obtained seem to indicate that quantum-mechanically revisited GR turns out to be capable of rectifying singularities, thereby facilitating accurate predictions across a wide range of scales.

The RGUP introduces a modification to the momentum operator p^{μ}, transforming the auxiliary four-momentum p_0^{μ} into $\phi(p_0)\, p_0^{\mu}$.[9–11] Here, the function $\phi(p_0)$ is defined as $1 + 2\beta p_0^{\rho} p_{0\rho}$, where β represents the RGUP parameter, p_0^{μ} represents the auxiliary four-momentum of a quantum particle and ρ is a dummy index. The indexes μ and ν range over the set $\{0, 1, 2, 3\}$. The RGUP presumably extends the uncertainties associated with quantum-mechanical operators by incorporating the effects of relativistic energies and gravitational fields. Therefore, RGUP provides a comprehensive extension of QM, which assists in reconciling the principles of QM and GR. In this regard, we emphasize that the metric tensor in the RGUP approach appears to be influenced by the momenta of the particles. Nevertheless, if one examines closely, it becomes clear that the definition of momentum itself necessitates a metric tensor, as demonstrated in Lagrangian analysis or in the Hamiltonian within a properly defined Klein–Gordon framework. As a result, it seems that nonlinear feedback arises, which would inhibit an analysis of any stable attractor. Hence, we believe that further analysis should be conducted to clarify how the momentum expectation can be achieved without a background metric tensor. Up to this point, we have opted for truncation at the linear order, which is based on an approximate linear perturbation theory, thus yielding the ability to analyze properties in a strong gravity regime, especially in proximity to singularities.

The first author (AT) developed the concept of generalized structures, either tangent Finsler or cotangent Hamilton manifolds, as an extension of conventional GR.[3–8,13,14] The incorporation of associated notions in quantum geometry, noncommutative algebra and duality-symmetry configurations is briefly described. As mentioned, this approach generalizes the four-dimensional Riemann geometry to a higher-dimensional geometry, which can be characterized as either Finsler geometry, incorporating space–time and tangent velocity space, or Hamilton geometry, which involves space–time and cotangent momentum space. The procedure for generalizing Finsler to Hamilton geometry is based on the positive homogeneity of the corresponding structure. The integration of the kinematic theory pertaining to a quantum particle with positive mass m facilitates the extension of the Finsler to Hamilton structure. This structure is defined by coordinates and tangent directions derived from velocities $F(x, \dot{x})$ and can be transformed into a Hamilton structure characterized by coordinates and cotangent directions associated with momenta $F(x, p)$. The property of positive homogeneity enables the expression $F(x, m\dot{x}) = mF(x, \dot{x})$, leading to the equivalence $F(x, m\dot{x}) \equiv F(x, p)$. With the auxiliary four-vectors, this can be re-expressed as $F(x_0^{\mu}, p_0^{\nu})$, i.e. as a space-space Hamilton structure. Now, the integration of RGUP and its quantum-mechanical elements into the structure $F(x_0^{\mu}, p_{0\nu})$ becomes achievable. The function $\phi(p_0)$ plays a crucial role in this integration, resulting in the formulation of the quantized structure $F(x_0^{\mu}, \phi(p_0)p_{0\nu})$. It is noteworthy that the expressions $F(x_0^{\mu}, \dot{x}_0^{\nu})$, $F(x_0^{\mu}, m\dot{x}_0^{\nu})$, $F(x_0^{\mu}, p_{0\nu})$ and $F(x_0^{\mu}, \phi(p_0)$ $p_{0\nu})$ all exhibit positive homogeneity. In this respect, there exists a dual nature between tangent and cotangent geometries. It is important to note, however, that the transformation from the tangent to the cotangent bundle is nonlinear. In this regard, it has been demonstrated that our approach, which relies on positive homogeneity, is equivalent to the Legendre transformation.[15–17]

The Hessian for the quantum-mechanically revisited squared Hamilton structure, $F^2(x_0^\mu, \phi(p_0)p_{0\nu})$, results in a metric tensor that encompasses higher dimensions and obviously quantum-mechanical ingredients. For simplicity, we assume $(x_0^\mu, p_{0\nu}) \equiv (x_0, p_0)$. Specifically, the quantized Hamilton structure comprises four-dimensional space–time and four-dimensional momentum space. The last significant step involves the derivation of the four-dimensional fundamental metric tensor for quantum-mechanically revisited GR. This process is based on establishing connections between the line elements present in both manifolds. Furthermore, under the assumption that the conventional fundamental metric tensor is fully maintained, the quantum-mechanically revisited fundamental metric tensor turns out to be expressed as a conformal transformation of its nonquantized version. The conformal transformation is not *ad hoc* proposed; it is rather a mathematically constrained formulation that arises from a thorough and rigorous derivation process. Consequently, this indicates that GR is set to be effectively expanded to include quantum (low) scales. In addition, quantum-mechanically revisited GR is expected to experience additional curvatures, which serve as additional sources for gravitation, along with quantum characteristics. This paper examines the impact of the proposed quantization on the space singularity in GR.

Investigation of the timelike geodesic congruence of the *simplest* spherically symmetric vacuum solutions of EFE with the quantum-mechanically revisited fundamental metric tensor, which presumably encodes all local geometric information of the associated curved space–time, is the main aim of this study. As previously stated, a similar analysis has been performed for other solutions of EFE.[3–8] By interpreting the trajectory congruence as flow lines produced by velocity fields,[18,19] an assessment regarding the regulation of the space singularity can be concluded. The Raychaudhuri equations may then be used to derive the flows, which can be either geodesic or nongeodesic vector fields.[20] As collapse generally follows a phase of negative expansion, the sign of the expansion evolution equation can be employed to determine the characteristics of the timelike geodesic congruence. A positive sign denotes a nonsingular congruence, whereas a negative sign indicates a singular one. In this respect, we point out that the Raychaudhuri equations, although they possess a geometric nature, do not seem to account for the backreaction of matter on the geodesic, while this paper does address this aspect. Therefore, the focusing theorems should be viewed as applicable to *test masses* rather than to the *masses that influence the geometry with their content*. It may still be necessary to modify the RGUP approach to reflect this consideration. This might also be another reason why the focusing theorem in this quantum approach requires additional fine-tuning. We stress that this work does *not* propose a full theory of quantum gravity. Rather, it develops a geometric quantization (or quantum-mechanically inspired augmentation) of the metric tensor based on phase-space constructions and RGUP-motivated operator deformations. Our results should therefore be viewed as suggestive and heuristic, indicating directions in which operator-level treatments and constraint analyses should be pursued. The analysis in this work is intentionally restricted in scope. We consider a fixed classical background geometry (Schwarzschild space–time) and probe the consequences of the conformally modified metric tensor on timelike radial geodesic congruence only. Rotating geometries, null congruence, and fully dynamical backreaction of the conformal factor on the space–time geometry are not addressed here. Moreover, the RGUP-induced modifications are treated perturbatively and within a mean-field approximation (MFA). These restrictions allow analytic control and transparent physical interpretation, while leaving more general extensions (e.g. Kerr geometry, null focusing, or self-consistent semiclassical dynamics) for future work.

This paper is organized into several sections. In Sec. 2, a brief description of the proposed canonical geometric quantization ansatz for the "*fundamental metric tensor*" is outlined, which relies on quantum mechanical revisions in the context of eight-dimensional geometry. In Sec. 3, the results are discussed. The analytical study of expansion evolution and the characteristics of space singularities is discussed in Subsec. 3.1. The numerical analysis is summarized in Subsec. 3.2. Conclusively, Sec. 4 presents the final conclusions.

2. Canonical Geometric Quantization Ansatz for Fundamental Metric Tensor

It was pointed out that the principles of GR and QM can be reconciled only if they are adequately generalized.[3–8,13,14] This requires that the influences of nonvanishing gravitational fields and relativistic energies can be incorporated into QM, and that Riemann geometry in GR can be broadened to include Finsler and Hamilton geometry.[3–8,13,14] A relativistic generalization shall be introduced to the Heisenberg Uncertainty Principle (HUP), the key fundamental theory of QM, turning HUP into RGUP.[9–11] In this work, we adopt the RGUP deformation of the auxiliary four-momentum of the schematic form

$$p^\mu \mapsto \phi(p_0)p_0^\mu, \quad \phi(p_0) = 1 + 2\beta p_0^\rho p_{0\rho} + \mathcal{O}(\beta^2), \tag{1}$$

where β is a small RGUP parameter (with dimensions set by the Planck scale in appropriate units). The scalar combination $p_0^\rho p_{0\rho}$ is Lorentz invariant (for our signature $p^\rho p_\rho = -m^2$ on shell) and hence $\phi(p_0)$ is a covariant scalar function of momentum. The RGUP above should be viewed as an operator deformation when one promotes p_0^μ to an operator $\hat{p}^\mu$; the precise operator ordering and choice of domain will affect higher-order corrections. For clarity, we highlight the following:

- The RGUP form above is not unique; it depends on the chosen operator representation. Our choice keeps the leading covariant scalar deformation and is sufficient for the leading corrections examined in this paper.

- In the on-shell limit $p_0^\rho p_{0\rho} \to -m^2$, the combination $2\beta p_0^\rho p_{0\rho}$ can be interpreted as modifying an effective mass term at leading order. We discuss this connection explicitly below and identify the relations among β, β_1, β_2 used elsewhere in the text.

As a result, QM undergoes a generalization that introduces the metric tensor. The corresponding Klein–Gordon (KG) equation can be expressed as[10,11]

$$[-\hbar^2\Box + 2\beta\hbar^4\Box^2 + m^2]\Psi(t_0, \vec{x}_0) = 0, \tag{2}$$

where the d'Alembertian is given by $\Box = \partial_\mu\partial^\mu$. Because of $\Box^2$, the quantum-deformed KG equation becomes a fourth-order differential equation. It is obvious that vanishing RGUP parameter, $\beta \to 0$, reduces the preceding expression to the standard KG equation, $[p^\mu p_\mu + m^2]\Psi(t_0, \vec{x}_0) = 0$. Furthermore, the RGUP introduces modifications to the Schrödinger equation. Let us start with the quantum-deformed Hamiltonian of a quantum particle with mass m in a potential field $V(x)$. In position space, this can be expressed as

$$H \simeq -\frac{\hbar^2}{2m}\nabla^2 - \frac{\beta}{m}\hbar^4\nabla^4 + V(x). \tag{3}$$

Then, the time-dependent Schrödinger equation can be constructed as follows:

$$i\hbar\frac{\partial}{\partial t_0}\Psi(t_0, \vec{x}_0) = \left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{\beta}{m}\hbar^4\nabla^4 + V(x)\right]\Psi(t_0, \vec{x}_0). \tag{4}$$

As concluded with Eq. (2), the highest order of the differentials in the quantum-deformed Schrödinger equation is fourth, ∇^4. For vanishing β, the conventional Schrödinger equation is fully retrieved. Equation (1) displays several RGUP parameters denoted as β, β_1, β_2 which appear in distinct approximations to the relativistic and nonrelativistic limits. At leading order, one may relate these parameters by scaling relations (depending on the precise RGUP model); for the purposes of this analysis, we adopt the minimal identifications

$$\beta_1 \simeq \beta + \mathcal{O}(\beta^2), \quad \beta_2 \simeq \beta + \mathcal{O}(\beta^2), \tag{5}$$

so that the KG and Schrödinger deformations are consistent truncations of a single underlying RGUP expansion, i.e. in Eqs. (2) and (4), the first-order expansions in Eq. (5) are utilized. We explicitly note that at leading order a contribution $2\beta p_0^\rho p_{0\rho}$ can be reinterpreted on shell as a shift of the mass parameter $m \to m_{\text{eff}}$:

$$m_{\text{eff}}^2 = m^2(1 + 2\beta p_0^\rho p_{0\rho}/m^2) \approx m^2(1 + \mathcal{O}(\beta)), \tag{6}$$

but we prefer to keep β explicit in the deforming function $\phi(p_0)$ to emphasize the covariant momentum dependence and to avoid hiding operator content inside the mass definition. If desired, an equivalent presentation that absorbs these corrections into a position-dependent effective mass may be adopted; this equivalence should be made explicit when comparing with other approaches.

As briefly introduced, the auxiliary four-vector of the momentum p_0^ν shall be deformed by $\phi(p_0)$, while the auxiliary four-vector of the coordinate x_0^μ remains invariant. In this respect, the "*gravitization of QM*", as termed by Penrose, is so far based on three different approaches. Besides the RGUP that was introduced by one of the authors (AT), there are also two other approaches:

- The nonspontaneous collapse model, which introduces an *ad hoc* mechanism to regulate the collapse of the wave function superposition and thereby modifies the Schrödinger equation. Gravity was proposed as the fundamental component of this mechanism.[21] This could be identified with the mass m that was likewise introduced in the RGUP framework.[9]
- The transformation of the classical metric tensor into a dynamic tensor was proposed by converting the Hilbert space scalar product into a dynamic one.[22] The introduction of general covariance and quantum gravity imposes geometry on Hilbert spaces by quantum mechanical axioms. One might regard this as an *ad hoc* integration of quantum gravity, which adds dynamics to the metric tensor. In contrast, the geometric quantization approach imparts dynamics to the metric tensor from fundamental principles.[3–8,13,14]

It is not only the RGUP approach and thus the proposed relativistic gravitizational revisions of QM that highlight the crucial role of the function $\phi(p_0)$. The function $\phi(p_0)$ is also essential for the entire procedure of quantization of GR, which, as emphasized, is based on the "*quantization of the fundamental metric tensor*". To this end, another generalization should be introduced to GR. This is the Finsler/Hamilton structure, which is characterized by the coordinates $x_0^\mu(\zeta^\mu)$ and directions of velocities/momenta of the spacelike tangent velocities $dx_0^\mu(\zeta^\mu)/d\zeta^\mu$, where ζ represents an arbitrary parameterization that guarantees that $x_0^\mu(\zeta^\mu)$ on the Finsler/Hamilton manifold and x_0^μ on the Riemann manifold coincide. This parametrization allows, among other things, that the line element on the Riemann manifold and the equivalent line element on the Finsler/Hamilton manifolds are presumably identical. The generalization of GR proceeds as follows:

- First, the degree-1 homogeneity of the Finsler structure allows the introduction of the positive mass of the quantum particles. Thus, $x_0^\mu(\zeta^\mu)$ and $m\, dx_0^\nu(\zeta^\nu)/d\zeta^\nu$ ultimately represent the properties of the generalized Finsler structure, i.e. $F(x_0^\mu(\zeta^\mu), \dot{x}_0^\nu(\zeta^\nu))$ and $F(x_0^\mu(\zeta^\mu), m\, \dot{x}_0^\nu(\zeta^\nu))$ are positively homogeneous of degree one in $\dot{x}_0^\nu(\zeta^\nu)$, where m is a positive real number.
- Second, the function $\phi(p_0)$ derived from the RGUP approach generalizes the continuous auxiliary momentum $p_0^\nu(\zeta^\nu) = m\, \dot{x}_0^\nu(\zeta^\nu)$ to the discretized/quantized momentum operator constructed as $\phi(p_0)p_0^\nu(\zeta^\nu)$ and thereby the Finsler structure

can be converted into a Hamilton structure.[a] Also here, both $F(x_0^\mu(\zeta^\mu), p_{0\nu}(\zeta_\nu))$ and $F(x_0^\mu(\zeta^\mu), \phi(p_0)\, p_{0\nu}(\zeta_\nu))$ are positively homogeneous of degree one and $\phi(p_0)$ is positively homogeneous of degree zero in $p_{0\nu}(\zeta_\nu)$.

- Third, the proposed parametrization ensures the identification of the line element across various manifolds, thus allowing the quantized line element to be expressed in a pseudo-Riemann manifold. The goal is to maintain four-dimensional Riemann space–time and all of the postulates of conventional GR. Consequently, it is necessary to translate the metric derived in the tangent or cotangent manifold to the Riemann manifold. The viable procedure is predicated on the idea that line elements are the same for every manifold. It is clear that this procedure introduces approximations that cannot be avoided at this time.

Throughout this paper, we use the following index and notation conventions. Greek indices $\mu, \nu, \ldots$ run over space–time components $\{0,1,2,3\}$ and Latin indices $i, j, \ldots$ denote purely spatial components when needed. When an index or symbol carries the subscript "0", for example x_0^μ, p_0^μ, we mean the auxiliary (classical or background) phase-space coordinates that are used in intermediate derivations; these objects are not dynamical operators themselves, but serve to distinguish the auxiliary objects used in the Finsler/Hamilton constructions from averaged or operator-valued quantities. Furthermore, we adopt natural units $c = \hbar = 1$ unless otherwise stated. The chosen metric signature is $(-,+,+,+)$ so that $p^\mu p_\mu = -m^2$ for an on-shell particle.

A few remarks about Hamilton space are now in order.[23] Hamilton space is characterized by an n-dimensional smooth manifold M equipped with a continuous structure function $H : T^*M \to \mathbb{R}$. On its cotangent bundle, the Hamilton structure is smooth on $T^*M\backslash\{0\}$. To simplify, we utilize Einstein summation, allowing the representation of the auxiliary coordinates $(x_0^\mu, p_{0\nu})$ henceforth as (x_0, p_0). The phase-space Hamilton metric, which is characterized by (i) four-dimensional space–time and (ii) four-dimensional momentum space, can be derived from the Hessian matrix of $H(x_0, p_0)$,

$$g_{\mu\nu}(x_0, p_0) = \frac{1}{2}\frac{\partial}{\partial p^\mu}\frac{\partial}{\partial p^\nu} H(x_0, p_0). \tag{7}$$

Nearly everywhere on $T^*M\backslash\{0\}$, $g_{\mu\nu}(x_0, p_0)$ is nondegenerate.

Now the metric on the Hamilton manifold and structure, presumably equivalent to $F^2(x_0, \phi(p_0)p_0)$, can be derived directly from the Hessian matrix of the discretized quantum-mechanically revisited Hamilton structure at a common point, for example

[a] Legendre transformation makes it mathematically possible to transfer Finsler into Hamilton geometry. On the other hand, the translation based on homogeneity properties yields the same outcomes as the Legendre transformation.[3]

x_0 or equivalently $x_0(\zeta)$,

$$\tilde{g}_{\mu\nu}(x_0, p_0) = \frac{1}{2}\frac{\partial}{\partial p_0^\mu}\frac{\partial}{\partial p_0^\nu} F^2(x_0, \phi(p_0)p_0). \tag{8}$$

The variables on which $\tilde{g}_{\mu\nu}(x_0, p_0)$ depends obviously refer to auxiliary space–time coordinate and momentum space components, respectively. For clarity, we spell out the index placement and the reduction map used in these equations. The cotangent bundle T^*M carries coordinates $(x_0^\mu, p_{0\nu})$ where x_0^μ are base coordinates (upper indices) and $p_{0\nu}$ are covectors (lower indices). With Einstein summation, the Hamilton metric is obtained from the Hessian on T^*M as

$$g_{\mu\nu}^{(\mathrm{H})}(x_0, p_0) = \frac{1}{2}\frac{\partial^2 H(x_0, p_0)}{\partial p^\mu \partial p^\nu} \tag{9}$$

and the reduction to a space–time metric uses the parameterization $\zeta \mapsto (x_0^\mu(\zeta), p_{0\nu}(\zeta))$ together with the identification of the auxiliary momentum components $p_\nu(\zeta)$ with the dynamical momentum of a quantum particle evaluated along the trajectory. In what follows, we adopt the convention that space–time metric indices are lowered/raised with $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, while phase-space components are written with explicit coordinate labels when needed to avoid ambiguity.

For the sake of simplicity, we propose one of the simplest Finsler metrics, the Klein metric on $\mathbb{B}^n(1)$,[24]

$$F^2(x_0, \phi(p_0)p_0) = \phi^2(p_0)\frac{|p_0|^2 - |x_0|^2|p_0|^2 + \langle x_0, p_0\rangle^2}{(1 - |x_0|^2)^2}, \tag{10}$$

where $|\cdot|$ and $\langle\cdot\rangle$ are the standard Euclidean norm and inner product on $\mathbb{R}^n$.

The condition $H \equiv 0$, i.e. reparametrization invariance, which is relevant for canonical quantization procedures, is discussed in Subsec. 2.1.

2.1. *Reparametrization invariance and* $H \equiv 0$

It is well known that a first-order homogeneous Lagrangian $L(x_0, \dot{x}_0)$ which is homogeneous of degree one in $\dot{x}_0$ possesses a reparametrization invariance. The reparametrization invariance implies that the naive canonical Hamiltonian constructed via the Legendre transform can vanish identically (see, e.g. Ref. 25 and the discussion in Ref. 26). This observation is relevant for canonical quantization procedures and must be addressed when one attempts to treat $F(x_0, \dot{x}_0)$ as a Lagrangian. In this work, we avoid the $H \equiv 0$ obstruction by the following pragmatic choices, which should be regarded as part of the geometric quantization ansatz:

(1) We employ the squared structure $F^2(x_0, p_0)$, which is homogeneous of degree two, as the effective Hamiltonian function on the cotangent bundle; $F^2(x_0, p_0)$ thus plays the role of a nondegenerate Hamiltonian density for the phase-space Hessian (cf. Eqs. (4) and (5)). This avoids the immediate vanishing of the

canonical Hamiltonian while keeping the original homogeneity information encoded in $F(x_0, p_0)$.

(2) We assume positive mass (positive homogeneity in m) for the probe quantum particle; physically, this corresponds to choosing trajectories for which the Legendre map is nonsingular. This is the same technical assumption adopted in Refs. 26 and 27 when L^n techniques are used to regularize reparameterization invariance.

(3) We note that a more systematic treatment, such as Dirac–Bergmann constraint analysis, may be developed and will be addressed elsewhere; here we make explicit the assumptions and limits in which $F^2(x_0, p_0)$ provides a consistent nondegenerate phase-space metric from which the Hessian is computed.

In this respect, we emphasize that the derivative of $\tilde{g}_{\mu\nu}(x_0, p_0)$, Eq. (8), is also valid for arbitrary higher-dimensional structures,

$$\begin{aligned}\tilde{g}_{\mu\nu}(x_0, p_0) = &\left(\phi^2(p_0) + \frac{2\kappa\phi(p_0)F^2(x_0,p_0)}{(p_0^0)^2}\right)g_{\mu\nu}\\ &+ \frac{4\kappa F^2(x_0,p_0)}{(p_0^0)^2}\left(2\phi(p_0) + \frac{\kappa F^2(x_0,p_0)}{(p_0^0)^2}\right)\ell_\mu\ell_\nu\\ &- \frac{4\kappa F^3(x_0,p_0)}{(p_0^0)^3}\left(2\phi(p_0) + \frac{\kappa F^2(x_0,p_0)}{(p_0^0)^2}\right)(\ell_\mu\delta_{0\nu} + \ell_\nu\delta_{0\mu})\\ &+ \frac{2\kappa F^4(x_0,p_0)}{(p_0^0)^4}\left(3\phi(p_0) + \frac{2\kappa F^2(x_0,p_0)}{(p_0^0)^2}\right)\delta_{0\mu}\delta_{0\nu},\end{aligned} \tag{11}$$

where

$$\ell_\mu = \frac{\partial}{\partial p_0^\mu} F^2(x_0, p_0). \tag{12}$$

The quantum-mechanically revisited fundamental metric tensor is derived in Subsec. 2.2.

2.2. *Quantum-mechanically revisited fundamental metric tensor*

Utilizing the proposed parametrization ζ, the coordinates properly represent both geometries alongside their respective metric tensors. Then, the quantum-mechanically revisited line element related to $\tilde{g}_{\mu\nu}(x_0, p_0)$ can be expressed as

$$d\tilde{s}^2|_{\texttt{Phase Space}} = \tilde{g}_{\mu\nu}(x_0, p_0)dx_0^\mu(\zeta)dx_0^\nu(\zeta), \tag{13}$$

where the phase-space coordinates are composed as (x_0^μ, p_0^ν). To address the inquiry regarding the quantum nature of $\tilde{g}_{\mu\nu}(x_0, p_0)$, it is crucial to recall that discretization is integral to the quantization process. $\tilde{g}_{\mu\nu}(x_0, p_0)$ is apparently discretized through RGUP. Such discretization is not the sole reason for the quantum classification. As shown in Eq. (11), $\tilde{g}_{\mu\nu}(x_0, p_0)$ is indeed represented in terms of quantum operators and relevant quantum constants. The line element corresponding to the quantum-mechanically revisited fundamental metric tensor, $\tilde{g}_{\mu\nu}$, is given as

$$d\tilde{s}^2|_{\texttt{Space-time}} = \tilde{g}_{\mu\nu} d\zeta^\mu d\zeta^\nu. \tag{14}$$

The concept presented in Eqs. (13) and (14) aims to establish a relationship between $\tilde{g}_{\mu\nu}(x_0, p_0)$ in Hamilton geometry and $\tilde{g}_{\mu\nu}$ in Riemann geometry, with the parametrization being of significant importance. By solving both Eqs. (13) and (14) for $\tilde{g}_{\mu\nu}$, we obtain[13,14]

$$\tilde{g}_{\mu\nu}(x_0, p_0) = g_{\mu\nu}(x_0, p_0)\left[\frac{\partial x_0^\mu(\zeta)}{\partial \zeta^\mu}\,\frac{\partial x_0^\nu(\zeta)}{\partial \zeta^\nu}\right] \tag{15}$$

$$= \left(\phi^2(p_0) + 2\frac{\kappa}{(p_0^0)^2}F^2(x_0, p_0)\right)\left[1 + \frac{\dot{p}_0^\mu \dot{p}_0^\nu}{\mathcal{F}^2}(1 + 2\beta p_0^\rho p_{0\rho})\right] g_{\mu\nu}$$

$$+ \left[\frac{\dot{p}_0^\mu \dot{p}_0^\nu}{\mathcal{F}^2}(1 + 2\beta p_0^\rho p_{0\rho})\right] d_{\mu\nu}(x_0, p_0). \tag{16}$$

where the normalization factor $\mathcal{F}$ could be expressed as $m\mathcal{A}$ and represents the maximal proper gravitational force and $\mathcal{A}$ stands for the maximum proper acceleration given by $\mathcal{A} = c^7/G\hbar$. The normalization factor used relies on the notion of a maximal proper acceleration $\mathcal{A}$ (or maximal proper force). The idea of upper bounds on acceleration and force has appeared in various guises in the literature: Caianiello and Brandt argued for maximal acceleration bounds[28,29]; the Born reciprocity principle and work by Gibbons and collaborators have also explored symmetry principles and bounds in phase space.[30,31] Two logical interpretations must be distinguished:

(1) *Fundamental bound:* $\mathcal{A}$ and hence $\mathcal{F}$ is a true upper bound, possibly set by quantum gravity/Planck scale physics. If this is the case, terms in Eq. (12) proportional to inverse powers of $\mathcal{F}$ lead to finite, nonvanishing corrections and can qualitatively affect the singularity structure.
(2) *Effective bound:* $\mathcal{A}$ is an effective/phenomenological scale whose value may be pushed arbitrarily high in principle. In the limit $\mathcal{F} \to \infty$, the extra terms in Eq. (16) that are suppressed by powers of $1/\mathcal{F}$ vanish, so that the conformal modification reduces to a simpler form. Thus, the physical conclusions depend sensitively on whether $\mathcal{A}$ is truly fundamental or effective.

In this context, $\mathcal{F}^2$ is applied to normalize the proper forces $\dot{p}_0^\mu \dot{p}_0^\nu$, i.e. the squared proper gravitational force. In deriving Eq. (16), it was assumed that the products of mixed derivatives vanish completely. Here, we also identify $\tilde{g}_{\mu\nu}$ as the quantum-mechanically revisited fundamental metric tensor in Riemann geometry. It is clear that this tensor undergoes discretization through the RGUP approach, which guarantees that the measurement of relevant quantum quantities and operators is no longer defined by arbitrary continuity. The minimal limits established serve to define the discretization and impose restrictions on their minimum values. References 9–11 provide insights into the noncommutation relations between distance and momentum, which, for example, result in a nonzero uncertainty. Under RGUP, the auxiliary four-vectors representing distance and momentum are transformed into relativistic, four-dimensional operators that are interconnected with space–time via the fundamental metric tensor.

Under significant approximations, the metric tensor $\tilde{g}_{\mu\nu}(x_0, p_0)$ may be represented as a conformal transformation of $g_{\mu\nu}$. As discussed, this necessitates that $g_{\mu\nu}$ completely vanishes. Evidently, this conformal transformation is a product of mathematical derivation, not simply an *ad hoc* assumption. Analyzing this relationship reveals that the entire conformal coefficient is formed from quantum operators and relevant quantities. This finding provides additional evidence for the quantum nature of $\tilde{g}_{\mu\nu}(x_0, p_0)$. Consequently, the spectrum of the fundamental metric tensor is not continuous, as the quantum operators associated with space–time and momentum space are quantum-mechanically revisited. Further information regarding the measurement uncertainty of x_0^μ and p_0^ν can be found in Refs. 9–11. It is important to emphasize that the approximation of $\tilde{g}_{\mu\nu}(x_0, p_0)$ is unavoidable as long as the metric $d_{\mu\nu}(x_0, p_0)$ is formulated in terms of $g_{\mu\nu}$. The metric tensor $d_{\mu\nu}(x_0, p_0)$ reads

$$d_{\mu\nu}(x_0, p_0) = \frac{6\kappa\phi(p_0)}{(p_0^0)^2} F^2(x_0, p_0) \left[\ell_\mu \ell^\sigma g_{\sigma\mu} - F(x_0, p_0)\ell^\sigma(\delta_{0\nu} g_{\sigma\mu} + \delta_{0\mu} g_{\sigma\nu}) + \frac{2+\phi(p_0)}{8\phi(p_0)} F^2(x_0, p_0)\delta_{0\mu}\delta_0^\sigma g_{\sigma\nu}\right], \tag{17}$$

where $\sigma \in \{0, 1, 2, 3\}$ is a free index,

$$\phi_\mu(p_0) = \frac{2\kappa F(x_0, p_0)}{(p_0^0)^3}(p_0^0 \ell_\mu - F(x_0, p_0)\delta_{0\mu}), \tag{18}$$

$$\phi_{\mu\nu}(p_0) = -\frac{4\kappa F(x_0, p_0)}{(p_0^0)^3}(\ell_\nu \delta_{0\mu} + \ell_\mu \delta_{0\nu}) + \frac{2\kappa F(x_0, p_0)}{(p_0^0)^2}\ell_\mu \ell_\nu + \frac{6\kappa F^2(x_0, p_0)}{(p_0^0)^4}\delta_{0\nu}\delta_{0\mu} + \frac{2\kappa F(x_0, p_0)}{(p_0^0)^2}\ell_{\mu\nu}, \tag{19}$$

$$\ell_{\mu\nu} = \frac{\partial}{\partial p_0^\mu}\ell_\nu. \tag{20}$$

It is likely that the truncation of $\tilde{g}_{\mu\nu}$, omitting the $d_{\mu\nu}(x_0, p_0)$-term, may reduce accuracy. However, since the quantum-mechanical operators and their related quantities are still to be approximated using the mean-field approach, for example, the effects of $d_{\mu\nu}(x_0, p_0) = 0$ could be deemed acceptable. Furthermore, in the absence of any other viable alternatives and to provide insights into the potential for geometric quantization, and thus the possible attenuation or regulation of space–time singularities, the proposed quantization of the metric tensor is fundamentally based on[13,14]

$$\tilde{g}_{\mu\nu}(x_0,p_0,\dot{p}_0) = \left(\phi^2(p_0) + 2\frac{\kappa}{(p_0^0)^2}F^2(x_0,p_0)\right)\left[1+\frac{\dot{p}_0^{\mu}\dot{p}_0^{\nu}}{\mathcal{F}^2}(1+2\beta p_0^{\rho}p_{0\rho})\right]g_{\mu\nu}$$
$$= C(x_0,p_0,\dot{p}_0)g_{\mu\nu}. \tag{21}$$

Here, $C(x_0,p_0,\dot{p}_0)$ is identified as the conformal coefficient, which is not an arbitrary construct. The arguments of C refer to the auxiliary quantities $x_0,p_0,\dot{p}_0$. It serves as an appropriate representation of rigorous mathematical derivatives. A necessary consistency requirement is that standard GR be recovered when quantum-mechanically inspired corrections are switched off. In the present framework, this corresponds to the simultaneous limits

$$\beta \to 0, \quad F(x_0,p_0) \to \infty, \tag{22}$$

for which $\phi(p_0) \to 1$ and all terms suppressed by inverse powers of $F(x_0,p_0)$ vanish. In this limit the conformal coefficient satisfies

$$C(x_0,p_0,\dot{p}_0) \to 1 \tag{23}$$

and the modified metric tensor $\tilde{g}_{\mu\nu}$ reduces identically to the classical space–time metric tensor $g_{\mu\nu}$. Consequently, the Raychaudhuri equation and geodesic focusing properties revert to their standard GR form. This provides a nontrivial internal consistency check of the construction. The conformal coefficient $C(x_0,p_0,\dot{p}_0)$ may be interpreted as an effective phase-space backreaction of quantum-inspired kinematics on the classical space–time geometry. Values $C(x_0,p_0,\dot{p}_0) > 1$ act as a local stretching of proper time and length scales along timelike trajectories, weakening geodesic convergence, while $C(x_0,p_0,\dot{p}_0) < 1$ enhances convergence. From this perspective, $C(x_0,p_0,\dot{p}_0)$ plays a role analogous to an effective geometric pressure generated by RGUP-induced momentum fluctuations, encoded here through the phase-space Hessian structure. Equation (21) emphasizes that the metric tensor $\tilde{g}_{\mu\nu}(x_0,p_0,\dot{p}_0)$ is fundamentally influenced by x_0, p_0 and $\dot{p}_0$. For the sake of convenience, we will henceforth denote it as $\tilde{g}_{\mu\nu}$. As noted, the only feasible method for estimating quantum operators and their quantities, including metrics, is through the use of averaging techniques, for example the MFA. The components of the metrics, for instance, will be defined through averaged values. This suggests that the averaging approach relies on operators rather than classical variables.[32] In this respect, it is assumed that quantum operators are used to define the states of space–time as a quantum system[33]

$$\langle\hat{x}(\hat{\xi})\rangle^{\mu} \equiv \langle x\rangle^{\mu}(\hat{\xi}), \tag{24}$$

where $\hat{\xi}$ represents quantum operators that manipulate the essential characteristics of space–time. It is essential to highlight that our estimation is fundamentally based on the expectation value $\langle\hat{x}(\hat{\xi})\rangle^{\mu}$, namely $x^{\mu}(\hat{\xi})$. For a specified index μ, a quantum operator is employed to determine the average coordinate, for example. In the context of metrics and analogous higher-dimensional operators, the additional index

ν reveals details about the dimensionality of that operator. In light of this discussion, one could assign various real values to $C(x_0, p_0, \dot{p}_0)$, at least for the sake of approximate numerical simulations.

Concerning the averaging approach, it should be emphasized that the Hessian quantity within the phase space which is likely subject to averaging necessitates the proposal of an initial state from which the averages are derived. The conformal coefficient $C(x_0, p_0, \dot{p}_0)$ is estimated here by applying a mean-field (expectation value) approximation to the relevant phase-space operators. This is an operational choice reflecting the measurement collapse and the need to extract $C(x_0, p_0, \dot{p}_0)$-number coefficients from operator expressions. We acknowledge that this averaging selects a particular representative from a family of possible expectation values and that a full operator treatment will generally produce distributions of outcomes; the qualitative numerical results reported here should therefore be interpreted within this approximation scheme. In the present calculations, this initial state is the corresponding classical metric. Specifically, the classical Hamilton metric represents the initial state. Moreover, given that this analysis appears to pertain to the motion of quantum particles, it remains essential to establish the duration for which the trajectory representation (geodesics) can be considered reliable. This specific issue could be discussed in a future work.

A few comments on the conformal transformation, Eq. (21), are now in order.

(1) First, it is known that a simple conformal or disformal transformation of the metric is insufficient to address the singularity problem.[34] The findings indicate that as long as the metric transformation involving either a scalar field[35] or a vector field[36] remains regular and invertible, the fundamental characteristics of the theory, including higher dimensions, will be maintained.[37]
(2) Second, the conformal transformation we present is not an imposed approach. It is a robust mathematical derivation that is nearly rooted in first principles. The coefficient related to the conformal transformation is not a trivial scalar or vector applied superficially; it represents a significant theoretical insight. Furthermore, this coefficient integrates quantum mechanical operators and quantities, whose effects remain evident despite any suggested average that would be implemented.

Now we give estimations for various parameters. It is possible to establish upper bounds on the RGUP parameter $\beta = \beta_0 G/(c^3\hbar) = \beta_0(\ell_p/\hbar)^2$ from table-top experiments[38] and astrophysical observations.[39,40] The physical quantities that determine β and its variations include the Planck constant $\hbar = 1.0545 \times 10^{-34}$ joule seconds, the Planck length $\ell_p = \sqrt{\hbar G/c^3} \simeq 1.616 \times 10^{-35}$ meters, the gravitational constant $G = 6.674 \times 10^{-11}$ cubic meters per kilogram per second squared and the speed of light in a vacuum $c \simeq 300,000$ kilometers per second. In this regard, one finds that the RGUP parameter β carries inverse momentum-squared dimensions, while $\mathcal{F} = m\mathcal{A}$ has dimensions of force. All terms entering the conformal coefficient $C(x_0, p_0, \dot{p}_0)$

are therefore dimensionless, as required for a consistent conformal rescaling of the metric.

Furthermore, and as mentioned earlier, the four-dimensional fundamental metric tensor on the Riemann manifold, $g_{\mu\nu}$, can be conformally scaled by $C(x_0, p_0, \dot{p}_0)$ whose phase-space dependence or even higher is evident, and despite the substantial approximation, it effectively incorporates notable quantum mechanical revisions. Despite its approximate nature, the conformal coefficient $C(x_0, p_0, \dot{p}_0)$ displays a well-defined structure and maintains its capacity to explore quantum-mechanically revisited GR. Its various components include:

- The squared proper force $\dot{p}_0^{\mu}\dot{p}_0^{\nu}$, where an overdot refers to a derivative with respect to the parametrization ζ.
- In the resulting curved space–time, $\mathcal{F}$ denotes the maximal proper force that accelerates a quantum particle with mass m as it falls freely during its gravitational motion.[13,14] In this regard, we refer to the maximal proper acceleration which was discovered by Caianiello $\mathcal{A} = 2\pi(c^7/G\hbar)^{1/2}$.[41,42] This is apparently related to the maximum proper gravitational force $\mathcal{F} = (m\mathcal{A})^{-1}$.
- The Klein metric F is defined on the Finsler/Hamilton manifold and has phase-space dimensions, i.e. four-dimensional space–time and four-dimensional momentum space, while $\tilde{g}_{\mu\nu}$ is defined locally on the Riemann manifold.
- In the case of the quantum particle, $\kappa = \beta(p_0^0)^2$, where $p_0^0 = E/c$ is the 0th component of the four-momentum and E is the energy of this particle.

Let us now elaborate on the potential differences between the Finsler and Hamilton metrics. This distinction evidently arises from the nonlinear relationship between the Finsler and Hamilton structures. Both metrics can be derived through the Hessian matrix associated with the respective structure. As previously discussed, while both structures are defined by the manifold, the Finsler structure is additionally characterized by a tangent bundle that relates to velocity fields, whereas the Hamilton structure pertains to a cotangent bundle that relates to momentum fields. Despite their inherent duality, the transformation from the Finsler structure to the Hamilton structure is nonlinear. This transformation can be achieved through the Legendre transformation. The current analytical study has utilized the properties of homogeneity. It has been demonstrated that the results obtained are consistent with those derived from the Legendre transformation.[3] For the sake of simplicity, we assumed that the Finsler metric can be expressed by the Klein metric on $\mathbb{B}^n(1)$.[24] Since this also preserves the same homogeneity property, we utilize it in Hamilton geometry.

We now address the auxiliary four-coordinate x_0^{μ} with particular emphasis on its 0th component, the time operator x_0^0. Let us start from Ref. 43, where the time operator was found to behave analogously to energy, i.e. it is bounded and self-adjoint. To clarify this point, let us assume that the time operator is canonically conjugate to the Hamiltonian H. This allows for a justification to believe that this pair of operators conforms to the time-energy canonical commutation relation $[T, H] = i\hbar$. Moreover, it seems to represent the Heisenberg equation of motion $dT/dt = [T, H]/i\hbar$.[43] This indicates that time operators evolve in steps of time t, which signifies that T fulfills the covariance requirement $T(t) = T(0) + t$ for all instances.[43,44] Considering the semibounded Hamiltonian in QM, we acknowledge, on one hand, the historical rejection of the existence of any such time operator.[45,46] As a non-self-adjoint, covariant, positive-operator-valued measure (POVM) observable, the time operator can, on the other hand, be reintroduced into QM, due to the essential finding that a quantum observable does not need to be self-adjoint POVM.[47] Lastly, it is crucial to acknowledge that the well-known claim made by Pauli[48] has been refuted by numerous counterexamples, which has resulted in an increased emphasis on the examination of self-adjoint time operators. This has enabled us to underscore that the premise of a self-adjoint operator that is canonically conjugate to a semibounded or discrete Hamiltonian does not conflict with Ref. 49. The claim that only non-self-adjoint time operators exist within single Hilbert quantum mechanics seems to lack universality. This study serves as an illustration of this concept.

Having briefly obtained the quantum-deformed metric tensor of GR and reviewed as well as examined nearly all related aspects, it is now appropriate to present an application of this work. The most fitting application would involve phenomena where traditional GR encounters difficulties. Our focus will be on the space singularity within black holes. The following section presents analytical and numerical examinations of the timelike geodesic congruence in the simplest spherically symmetric vacuum solution of EFE, the Schwarzschild solution, with both conventional and quantum-deformed metric tensors of GR. This section aims to provide a comprehensive exploration of the space singularity, encompassing an examination of both versions of the fundamental metric tensors.

3. Implication to Expansion Evolution and Space Singularity

3.1. *Analytical analysis*

After deriving the quantum-mechanically revisited fundamental metric, namely the four-dimensional $\tilde{g}_{\mu\nu}$ which characterizes the Riemann geometry, the squared line element $-d\tilde{s}^2$ can be utilized to calculate the squared proper time $d\tilde{\tau}^2$. It is essential to emphasize that this analytical analysis is grounded in the framework of $\tilde{g}_{\mu\nu}$, specifically within Riemann geometry. The mathematical derivatives executed in the context of phase-space geometry, whether Finsler or Hamilton, are intended to incorporate quantum-mechanical modifications into the corresponding metric. Then, this metric can be expressed in terms of the four-dimensional $\tilde{g}_{\mu\nu}$, i.e. as a conformal transformation. With this foundation established, the analytical analysis will be restricted to Riemann geometry. Subsequently, the geodesic equations along the path of the line element in curved space–time can be obtained (see C). We define the expansion scalar Θ for a timelike congruence with unit tangent u^μ by

$$\Theta \equiv \nabla_\mu u^\mu = u^\mu_{;\mu}, \tag{25}$$

which measures the local volume rate of change of a bundle of geodesics. For a spherically symmetric radial congruence, i.e. the Schwarzschild solution of EFE, one can write, for the radial component u^r,

$$\Theta = \frac{1}{r^2}\partial_r(r^2 u^r) \tag{26}$$

and the Raychaudhuri evolution is given by (C)

$$\Theta = \pm\frac{3}{\sqrt{2}}\frac{1}{r}\sqrt{\frac{M}{r^3}}, \tag{27}$$

$$\frac{d\Theta}{d\tau} = -\frac{9}{2}\frac{M}{r^3}, \tag{28}$$

which we quote for reference (see C).

The expansion Θ can be determined from the four-velocity, which can be approximated using the four geodesic equations and the four-velocity normalization condition, such as $\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1$ (A and C). In the case of $g_{\mu\nu}$, the nonquantized fundamental metric tensor, the expansion $\Theta = u^\mu_{;\mu}$ is defined in the simplest spherically symmetric vacuum solution of EFE, the Schwarzschild metric. The subscript $;\mu$ denotes covariant derivatives or semicolon derivatives with respect to x^μ. Semicolon derivatives represent connections on the tangent bundle that specify the derivatives along the tangent vectors on the manifold. Alternatively, $u^\mu_{;\mu}$ can be substituted with $u^\mu_{,\mu} + u^\sigma\Gamma^\mu_{\sigma\mu}$, where the comma derivative $,\mu$ represents the partial derivative with respect to x^μ. The affine *linear* connections on the manifold, namely $\Gamma^\mu_{\sigma\mu}$, allow the discrimination of tangent vector fields, even when they linearly link adjacent tangent spaces. In other words, the affine *linear* connections ensure that the tangent vector fields remain on the manifold.

The generalized normalization condition, $\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1$, makes use of geometrical units where $G = c = 1$, where, among other things, the speed of light in vacuum c and the gravitational constant G are set to unity. B provides a detailed explanation of how this normalization condition not only determines the corresponding geodesic equations but also ensures that the conservation laws are fulfilled (A). It is hypothesized that the normalization condition presents an affine parametrization of the timelike geodesic congruence. This affine parametrized congruence is necessary in order to have a unit vector field, which is required by the projection operator. The Lagrangian of the quantum particle can be represented as $g_{\mu\nu}u^\mu u^\nu$ or $\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu$ for the conventional and quantum-mechanically revisited fundamental metric tensors, respectively. As a result, the Euler–Lagrange equations will yield positive, negative, or zero values for timelike, spacelike, or null geodesics, respectively.

Based on the quantum-revisited fundamental metric tensor, the expansion has been quantum-mechanically reevaluated as $\tilde{\Theta} = \tilde{u}^{\mu}_{;\mu} = \tilde{u}^{\mu}_{,\mu} + \tilde{u}^{\sigma}\tilde{\Gamma}^{\mu}_{\sigma\mu}$. Alternatively, $\tilde{\Theta}$ can be calculated as $r^{-2}(r^2 u^r)_{,r}$, where r represents the radial distance and u^r the corresponding velocity component. Now, it is in order to review the phenomenology related to the sign of the expansion, Θ.

- A positive Θ refers to an outgoing geodesic and the corresponding congruence is diverging.
- A negative Θ is just the opposite, i.e. the geodesic is ingoing and the congruence is converging or collapsing.

A family of curves (geodesics) is defined by a congruence in an open subset $O \subset M$, where M represents the manifold. At each point $p \in O$ as it evolves over time, only one curve (geodesic) passes through it, with no intersections between any two trajectories. The expansion quantifies the isotropic change in volume that accommodates these curves (geodesics), while the curves themselves exhibit fluctuations. Recent literature has discussed geodesic congruence in Raychaudhuri equations and space–time which is enhanced by quantum effects.[50,51]

The derivation of the evolution equation for the expansion is straightforward: $d\Theta/d\tau = (d\Theta/dr)(dr/d\tau)$. It is evident that $dr/d\tau$ represents the r-component of the velocity. The focusing theorem explains the negative evolution of expansion:

- a diverging congruence with $\Theta > 0$ diverges less along the forthcoming evolution, while
- a converging congruence with $\Theta < 0$ converges more rapidly, i.e. collapses.

Based on the physical interpretation of these findings, it is likely that the geodesics are being focused as a result of the strong gravitational force. The change in space–time near a region of mass over time, as recognized by an observer moving with the flow, can be determined by utilizing the evolution equation of expansion, specifically the rate of expansion of the timelike geodesic congruence. In certain scenarios, the derivative of expansion with respect to proper time τ, equivalent to cosmological time t, may be either negative or positive. A negative value indicates a contraction in expansion, while a positive value indicates further expansion. This analysis belongs to the movement around a mass region, known as the Raychaudhuri equations, which serve as the fundamental basis for the Penrose–Hawking singularity theorems.[52,53] Whether m is interpreted as the test mass in the context of the Raychaudhuri equations or as the mass that affects the geometry with its content will be discussed in further detail in another context. The mass symbol m must differentiate between the quantum particle discussed in the previous section and the mass associated with the Schwarzschild solution referenced in this section.

3.2. *Numerical analysis*

Figure 1 illustrates a numerical analysis of the evolution function of the expansion using $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, namely $d\Theta/d\tau$ and $d\tilde{\Theta}/d\tau$, respectively, as functions of the radial distance r and the approximate conformal coefficient $C(x_0, p_0, \dot{p}_0)$. The coefficient $C(x_0, p_0, \dot{p}_0)$ arises from the proposed geometric quantization approach, Eq. (21).

The numerical results of the expansion evolution $d\Theta/d\tau$ are contrasted with the difference between $d\tilde{\Theta}/d\tau$ and $d\Theta/d\tau$, which represents the pure quantum contributions to expansion evolution. While the sign of the evolution expansion for $g_{\mu\nu}$ obviously remains negative everywhere, particularly at small r (black lattice), the sign of the evolution expansion for $\tilde{g}_{\mu\nu}$ seems to change its negativity at $C(x_0, p_0, \dot{p}_0) < 1$ to positivity at large $C(x_0, p_0, \dot{p}_0) > 1$ (blue lattice). It is evident that the transition from a positive to a negative sign maintains its continuity, even when r approaches unity.

A few remarks are now in order. At the point where $C(x_0, p_0, \dot{p}_0) = 1$, the quantum effects that impact the fundamental metric tensor and consequently the evolution of the expansion completely disappear, i.e. no quantization is imposed on $g_{\mu\nu}$. The question of whether the expansion evolution at this specific value differs from the expansion evolution associated with the conventional fundamental metric tensor can be approached from a phenomenological perspective. No noticeable convergence of both lattices is found at $C(x_0, p_0, \dot{p}_0) = 1$. This is nothing but a prompt sign change. It is evident that even minimal variations in $C(x_0, p_0, \dot{p}_0)$,

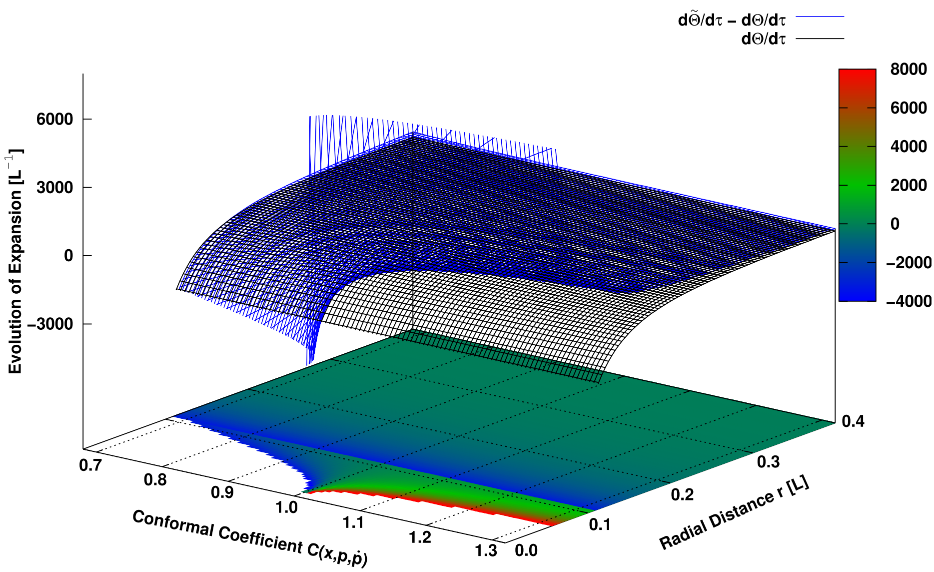

Fig. 1. (Color online) In geometrical units, the expansion evolution in the Schwarzschild solution of GR is illustrated as a function of radial distance r and the conformal coefficient resulting from the proposed geometric quantization $C(x_0, p_0, \dot{p}_0)$. The expansion evolution involving $\tilde{g}_{\mu\nu}$ (illustrated by the blue lattice) is contrasted with that involving $g_{\mu\nu}$ (depicted by the black lattice).

whether higher or smaller than unity, seem to have a substantial impact. This observation is confined to small r. With increasing values of $C(x_0, p_0, \dot{p}_0)$ beyond unity, the positive sign of $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ not only increases but apparently overcomes the negativity which is associated with decreasing r. We also find that the increase in positivity of $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ seems to have a nonlinear dependence on $C(x_0, p_0, \dot{p}_0)$. A similar pattern is also observed for values of $C(x_0, p_0, \dot{p}_0)$ that are less than one. With decreasing $C(x_0, p_0, \dot{p}_0) < 1$, the negativity of $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ increases so that the negativity associated with smaller r is enhanced. The third remark focuses on the most probable values of $C(x_0, p_0, \dot{p}_0)$. From the analysis of Eq. (21), it is evident that almost all terms and quantities constituting $C(x_0, p_0, \dot{p}_0)$ are likely positive. As a result, the absolute value of $C(x_0, p_0, \dot{p}_0)$ is anticipated to be greater than one. Thus, we arrive at the conclusion that the most probable implication of the proposed quantization is associated with a positive expansion evolution or a gradual attenuation of the space singularity. To conclude, the lower values of $C(x_0, p_0, \dot{p}_0)$ which produce negative $d\tilde{\Theta}/d\tau$ and $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ are indicative of space singularity, while the higher values corresponding to positive $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ suggest the absence of space singularity. Consequently, the values of $C(x_0, p_0, \dot{p}_0)$ seem to influence the presence and removal of space singularity from the Schwarzschild solution of GR.

In this regard, it should be noted that the scalar values assigned to the conformal coefficient are derived from an averaging process, such as MFA. It is important to emphasize that the qualitative numerical analysis is largely influenced by such averaging methodology, which is generally deemed acceptable, especially considering the truncation which has been imposed on deriving the fundamental metric tensor, Eq. (21). The proposed averaging method, although seemingly biased towards a specific instance from the continuous spectrum range of quantum operators, would not change the overall qualitative numerical analysis. This ensures that coherence in the conformal coefficient is maintained. This particular process emphasizes the widely recognized "*measurement problem*" in QM.[54] Nevertheless, it is an inevitable consequence of the collapse of quantum characteristics that occurs during the measurement process. Our proposed generalization of QM takes into account the influences of finite gravitational fields and relativistic energies. This feature corresponds with the physical mechanism that was earlier presented to create a control mechanism for the nonspontaneous collapse of quantum superposition.[21] By employing this analogy, we seek to illustrate that the "*measurement problem*" might be, if only to some extent, effectively resolved. Furthermore, it should be noted that the present calculations are founded on assumed values for $F^2(x_0, p_0)_{,\gamma}$ and M, both of which are assigned the value 0.95. Both values represent *ad hoc* inputs.

In Fig. 1, the black and blue lattices represent $d\Theta/d\tau$ and $d\tilde{\Theta}/d\tau - d\Theta/d\tau$, respectively. The colorful bar on the right displays the numerical values of both quantities projected onto the bottom plane. The focus here is on identifying where positive and negative expansion evolution is obtained. For the conventional fundamental metric tensor of GR, we observe that the expansion evolution is negative for all values of the radial distance (black lattice). As the radial distance decreases, the negativity of the expansion evolution grows rapidly. Based on the focusing theorem, we conclude that the negative evolution of expansion implies the presence of quickly converging congruence and continuous geodesics. This result supports the hypothesis that the spherically symmetric vacuum Schwarzschild solution of EFE exhibits a space singularity. It can be observed that the expansion evolution undergoes a rapid transition between outgoing (diverging) and ingoing geodesic (converging congruence). It can be concluded that the expansion evolution promptly changes between less divergence and rapid convergence. This phenomenon is indicated by the strong focusing, which implies a strong gravitational attraction. In the vicinity of a cosmic substance, the curvature of space–time swiftly changes between open and closed configurations.

The classical Hawking–Penrose singularity theorems rely on several assumptions, including the existence of a classical space–time metric tensor, standard energy conditions, and geodesic focusing governed by the Raychaudhuri equation. In the present framework, singularity attenuation does not arise from violating the energy conditions directly, but rather from a modification of the effective space–time geometry itself via the conformal factor $C(x_0, p_0, \dot{p}_0)$. The resulting change in the expansion scalar Θ modifies the focusing behavior of timelike congruence even when the classical energy conditions hold. In this sense, the attenuation mechanism operates by modifying the geometric premises of the singularity theorems rather than their matter-sector assumptions. By attenuation, we refer to the notion that the space singularity is considered to be "*eliminated*" or even "*controlled*" through the proposed geometric quantization approach. Upon quantifying the quantization by means of the conformal coefficient, one discovers that even a partial quantization can "*diminish*" the occurrence of singularity. Consequently, we have characterized this phenomenon as "*attenuation*", which implies a reduction in strength with lesser quantization, and so forth.

The analysis which is based on the quantum-mechanically revisited fundamental metric tensor emphasizes that the evolution of the geodesic congruence expansion displays a nonmonotonic correlation with the proposed quantization, a relationship that is extensively averaged in the conformal coefficient. A comparison between the pure quantum contribution, represented by the difference $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ (blue lattice) and $d\Theta/d\tau$ reveals a more intricate arrangement in the former (black lattice). Evidently, for small radial distances, the pure quantum contribution displays a gradual oscillation pattern, switching between negative and positive signs in response to the change of the averaged conformal coefficient. According to this numerical analysis, it can be suggested that (i) $C(x_0, p_0, \dot{p}_0) > 1$ is likely to regulate (attenuate) the singularity, while (ii) $C(x_0, p_0, \dot{p}_0) < 1$ tends to amplify the singular behavior. Therefore, the characteristic value $C(x_0, p_0, \dot{p}_0) = 1$ clearly denotes a saddle point, indicating that at this particular value, the impacts of the proposed quantization on either regulation or attenuation disappear so that the impacts of

$d\Theta/d\tau$ turn to become dominant. We conclude that the existence of space singularity seems to be greatly determined by the quantization that is imposed on GR.

Let us now review the physical units relevant to the numerical analysis shown in Fig. 1. The timelike geodesic congruence in the Schwarzschild space–time, which is the tensor $\nabla_\nu u_\mu$, does not have a single physical unit,

$$B_{\mu\nu} = \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{29}$$

Instead, each quantity linked to it has its own specific units: (i) the four-velocity field u^μ is dimensionless in geometric units with $G = c = 1$, or has units of [length/time^2] in SI; (ii) the expansion $\Theta = \nabla_\mu u^\mu$ can be defined in units of inverse proper time; (iii) the shear $\sigma_{\mu\nu}$ is also measured in units of inverse proper time; (iv) the vorticity $\omega_{\mu\nu}$ can similarly be expressed in units of inverse proper time, while the acceleration $a^\mu = u^\nu\nabla_\nu u^\mu$ is quantified in units of inverse length in geometric units or length/time^2 in SI. This results in a dimensionless congruence, whereas the kinematical quantities derived from it possess physical units. The inquiry regarding the physical units of expansion and its evolution is relevant, as it allows for the demonstration of where geometry begins to exhibit physical characteristics. Initially, the expansion itself is a scalar that quantifies the fractional rate of change of an infinitesimal volume element that is comoving with the geodesics, i.e. $(1/V)(dV/d\tau)$. Consequently, $[\Theta] = \mathtt{time}^{-1}$, specifically, the inverse of proper time or inverse length in geometrical units. The evolution is derived from $d\Theta/d\tau = u^\mu\nabla_\mu\Theta$, which measures the rate of change of the expansion itself along the geodesics, indicating how rapidly the local volume expansion (or contraction) is accelerating or decelerating; this can be assessed using the Raychaudhuri equation. Therefore, the physical units of $d\Theta/d\tau$ are time^{-2} (or length^{-1} in geometrical units). Figure 1 displays numerical results under the assumption of geometrical units. Thus, the evolution of expansion is expressed in length^{-1}, while the conformal coefficient $C(x_0, p_0, \dot{p}_0)$, as outlined in Eq. (21), is dimensionless, whereas the dimension of radial distance is evidently length or proper time.

4. Conclusions

A comprehensive examination of the timelike geodetic congruence in the simplest solution of the EFE is presented. This study mainly focuses on a concise derivation of the quantum-deformed metric tensor of GR, detailing most aspects associated with it, and subsequently applying this to the vacuum solution that exhibits spherical symmetry, the Schwarzschild metric. The results obtained are compared between the conventional, $g_{\mu\nu}$ and quantum-deformed fundamental metric tensor, $\tilde{g}_{\mu\nu}$. The latter refers to the four-dimensional tensor in Riemann geometry, which is derived from two interdependent generalizations and a new kinematic theory of quantum particles on the phase-space manifold. A detailed discussion of the derivation of $\tilde{g}_{\mu\nu}$ has been briefly incorporated to make this paper self-contained. More specifics can be recalled from Ref. 3.

One of the generalizations aims to introduce finite gravitational fields and relativistic energies into QM. This is achieved through the RGUP, which incorporates the function $\phi(p_0)$, whose roles extend beyond this specific generalization. The second generalization expands the four-dimensional Riemann geometry to Finsler/Hamilton, i.e. eight-dimensional tangent/cotangent geometry. The Finsler structure is then discretized using a kinematic theory of a quantum particle with positive mass. Similar to the Legendre transformation, the homogeneity property of the Finsler structure enables its mathematically accurate mapping into a Hamilton structure, where quantum deformation is properly imposed, namely that associated with RGUP. The Hessian matrix is then utilized to derive a metric on phase-space, i.e. four-dimensional space–time and four-dimensional momentum space. By employing the geometric quantization ansatz, the four-dimensional quantum-mechanically revisited metric tensor can be derived by adjusting the measures of the line element in both Finsler/Hamilton and Riemann space. This facilitates the transformation of the phase-space metric into a four-dimensional Riemann metric. The underlying motivation for this transformation is to maintain all postulates of GR. A subsequent development that elaborates on one of the postulates includes the expansion of Riemann geometry and the corresponding space–time and metric tensor. This indicates the intention to relax the limitation of the four-dimensional Riemann metric to an eight-dimensional metric.[55]

The focusing theorem postulates that positive expansion is associated with a diverging congruence and is expected to diverge to a lesser extent. Conversely, negative expansion is related to a converging congruence and is likely to converge rapidly during subsequent evolution. The conventional fundamental metric tensor reveals that the expansion evolution is consistently negative across all points. This observation leads to the inference that space–time is probably characterized by converging congruence and continuous geodesics, indicating a space singularity. The strong focusing effect suggests a strong gravitational attraction, causing the space–time to exhibit (i) open and closed curvature near cosmic substance and (ii) rapid transitions between outgoing (diverging) and ingoing geodesic (converging congruence) paths.

The "*measurement*" process in a compilation involving the inevitable wave function collapse is accompanied by a mean-field approximation, which focuses on a specific instance within the continuous spectrum range of quantum operators. This approximation enables the calculation of the expansion evolution deduced from the Raychaudhuri equations. It is observed that the quantum-deformed metric tensor governs the behavior of the geodesic congruence expansion, which becomes non-monotonic as the quantization varies. As shown in Eq. (21), the quantization is wholly contained in the conformal coefficient. Specifically, at small radial distances, the quantum contribution to the expansion evolution exhibits a rapid switch between negative and positive signs. The occurrence of a negative sign is indicative of the existence of a space singularity, whereas the positive sign suggests the regulation of the space singularity. The existence or nonexistence of the space singularity can be greatly influenced by even a small change in the proposed quantization, causing it to deviate from its characteristic unity. For an *average* conformal coefficient below unity, the likelihood of space singularity occurrence is heightened. Conversely, for an average conformal coefficient above unity, the space singularity weakens gradually and may even cease to exist entirely. This provides the basis for concluding that the proposed geometric quantization approach obviously contributes to controlling or potentially diminishing the space singularity.

Last but not least, let us highlight that the conformal transformation that related the conventional metric tensor to its quantized version is a product of mathematical derivation, not simply an *ad hoc* assumption. Furthermore, although the Finsler/Hamilton metric discussed in this paper is based on a tangent/cotangent manifold, it is still four-dimensional. The derivation of the phase-space metric will be explored in a future work. We note that this work focuses exclusively on timelike congruence. An analogous analysis for null congruence would require additional care, as the normalization of the tangent vector and the definition of expansion differ in the null case. Whether RGUP-induced conformal modifications can similarly affect null focusing and causal structure is an open question and will be explored elsewhere.

Appendix A. Proper Time and Geodesic Equations

For the quantum-mechanically revisited metric tensor $\tilde{g}_{\mu\nu}(x^\mu)$, the proper time can be expressed as

$$\tilde{\tau}_{ab} = \int_0^1 \sqrt{-\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} = \int_0^1 L\left[\frac{dx^\mu}{d\sigma}, x^\mu\right] d\sigma, \tag{A.1}$$

where L is the Lagrangian. Since the Euler–Lagrange equations can be derived using variational methods, we express them as

$$-\frac{d}{d\sigma}\frac{\partial L}{\partial(dx^\gamma/d\sigma)} + \frac{\partial L}{\partial x^\gamma} = 0. \tag{A.2}$$

These two terms can be obtained as follows:

$$\begin{aligned}\frac{\partial L}{\partial x^\gamma} = -\frac{L}{2}\Bigg\{ & C(x_0, p_0, \dot{p}_0)\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ & + g_{\mu\nu}\frac{2\kappa}{(p_0^0)^2}[1 + \mathcal{F}^{-2}(1 + 2\beta p_0^\rho p_{0\rho})]F^2(x_0, p_0)_{,\gamma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Bigg\},\end{aligned} \tag{A.3}$$

$$\frac{d}{d\sigma}\frac{\partial L}{\partial(dx^\gamma/d\sigma)} = -L\left[\tilde{g}_{\mu\gamma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\left(\frac{\partial\tilde{g}_{\mu\gamma}}{\partial x^\nu} + \frac{\partial\tilde{g}_{\gamma\nu}}{\partial x^\mu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]. \tag{A.4}$$

Thus, the geodesic equations read

$$\frac{d^2x^\mu}{d\tau^2} + \tilde{\Gamma}^{\,\mu}_{\delta\nu}\frac{dx^\delta}{d\tau}\frac{dx^\nu}{d\tau} + \frac{g_{\delta\nu}}{2\tilde{g}_{\mu\gamma}}F^2(x_0,p_0)_{,\gamma}C(x_0,p_0,\dot{p}_0)\frac{dx^\delta}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{A.5}$$

The essential impacts of the proposed quantization appear in the quantum-mechanically revisited affine connections. It is obvious that the quantum-mechanically revisited affine connections combine the conventional affine connections with additional contributions,[13,14]

$$\tilde{\Gamma}^{\,\mu}_{\delta\nu} = \Gamma^{\,\mu}_{\delta\nu} + \frac{F^2(x_0,p_0)_{,\gamma}}{2C(x_0,p_0,\dot{p}_0)}[\delta^{\,\mu}_{\nu} + \delta^{\,\mu}_{\delta} - g^{\mu\gamma}g_{\delta\nu}]. \tag{A.6}$$

This finding is essential for this study. Their effects on the generalization of GR are so significant that they cannot be exhaustively listed. It is now obvious that the construction of quantum-mechanically revisited GR is predicated on $\tilde{g}_{\nu\mu}$, which consequently gives rise to $\tilde{\Gamma}^{\,\mu}_{\delta\nu}$.

Appendix B. Generalized Normalization Condition and Conservation Laws

For the quantum-mechanically revisited metric tensor expressed as

$$\tilde{g}_{\mu\nu} = C(x_0,p_0,\dot{p}_0)g_{\mu\nu}, \tag{B.7}$$

the appropriate normalization condition can be given as

$$C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\mu u^\nu = -1. \tag{B.8}$$

This feature must facilitate the extraction of the quantum-mechanically revisited geodesic equations. Let us begin by considering the covariant differentiation

$$\begin{aligned} &C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\mu u^\nu\;\tilde{\nabla}_\gamma + C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\mu\;\tilde{\nabla}_\gamma u^\nu + C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\nu\;\tilde{\nabla}_\gamma u^\mu \\ &+\; C(x_0,p_0,\dot{p}_0)u^\mu u^\nu\;\tilde{\nabla}_\gamma g_{\mu\nu} = 0, \end{aligned} \tag{B.9}$$

where $\tilde{\nabla}_\gamma u^\mu = u^\mu_{,\gamma} + u^\sigma\tilde{\Gamma}^{\,\nu}_{\sigma k}$. Through the elimination of the last terms in Eq. (B.9), we can derive

$$\begin{aligned} C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\mu u^\nu\;\tilde{\nabla}_\gamma + 2C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\mu\;\tilde{\nabla}_\gamma u^\nu &= 0, \\ 2C(x_0,p_0,\dot{p}_0)g_{\mu\nu}u^\mu\;\tilde{\nabla}_\gamma u^\nu &= -\,g_{\gamma\kappa}u^\gamma u^\kappa\;\tilde{\nabla}_\gamma C(x_0,p_0,\dot{p}_0), \\ 2u^\mu\;\tilde{\nabla}_\gamma u^\nu &= -\frac{g_{\gamma\kappa}}{\tilde{g}_{\mu\nu}}u^\gamma u^\kappa\;\tilde{\nabla}_\gamma C(x_0,p_0,\dot{p}_0). \end{aligned}$$

Consequently, the geodesic equations can be formulated as

$$\frac{d^2x^\mu}{d\tau^2} + \tilde{\Gamma}^{\,\mu}_{\gamma\kappa}\frac{dx^\gamma}{d\tau}\frac{dx^\kappa}{d\tau} + \frac{g_{\gamma\kappa}}{\tilde{g}_{\mu\nu}}C(x_0,p_0,\dot{p}_0)_{,\gamma}\frac{dx^\gamma}{d\tau}\frac{dx^\kappa}{d\tau} = 0. \tag{B.10}$$

It is obvious that Eqs. (B.10) and (A.5) are identical. Therefore, we conclude that the geodesic equations can be derived from the proposed generalized normalization condition, thereby demonstrating adherence to the principles of conservation laws.

Appendix C. Timelike Geodesic Congruence with $g_{\mu}\nu$

In natural units, the curved space–time representation of the simplest spherically symmetric vacuum solution of EFE, the Schwarzschild metric, can be expressed as[56]

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{C.11}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the two-sphere, at a constant radial distance r. Here, M represents the spherical mass around which EFE are solved. $F(r) = (1 + r_s/r)$, where $r_s = 2M$. The expansion function and its subsequent evolution are presented as follows:

$$\Theta = \pm\frac{3}{\sqrt{2}}\sqrt{\frac{M}{r^3}}, \tag{C.12}$$

$$\frac{d\Theta}{d\tau} = -\frac{9}{2}\frac{M}{r^3}. \tag{C.13}$$

Appendix D. Timelike Geodesic Congruence with $\tilde{g}_{\mu}\nu$

For $\tilde{g}_{\mu\nu}$, the quantum-mechanically revisited geodesic equations can be derived as

$$\frac{d\tilde{u}^t}{d\tau} = \left[\frac{2M}{r(2M-r)} - \frac{2F^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right]\frac{dx^t}{d\tau}\frac{dx^r}{d\tau}, \tag{D.14}$$

$$\begin{aligned}\frac{d\tilde{u}^r}{d\tau} = &\left[\frac{M(2M-r)}{r^3} + \frac{F^2(x_0,p_0)_{,\gamma}}{2C(x_0,p_0,\dot{p}_0)}\left(\frac{r-2M}{r}\right)^2 - \frac{F^2(x_0,p_0)_{,\gamma}}{2}\right]\left(\frac{dx^t}{d\tau}\right)^2\\
&+ \left[\frac{M}{r^2-2M} + \frac{7F^2(x_0,p_0)_{,\gamma}}{2vC(x_0,p_0,\dot{p}_0)} - \frac{F^2(x_0,p_0)_{,\gamma}}{2}\right]\left(\frac{dx^r}{d\tau}\right)^2\\
&+ \left[(r-2M) + \frac{F^2(x_0,p_0)_{,\gamma}}{2C(x_0,p_0,\dot{p}_0)}r(2M-r) + \frac{F^2(x_0,p_0)_{,\gamma}}{2}r(r-2M)\right]\left(\frac{dx^\theta}{d\tau}\right)^2\\
&+ \left[(r-2M)\sin^2\theta + \frac{F^2(x_0,p_0)_{,\gamma}}{2C(x_0,p_0,\dot{p}_0)}r(2M-r) + \frac{F^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}r(r-2M)\right]\\
&\times\left(\frac{dx^\phi}{d\tau}\right)^2,\end{aligned} \tag{D.15}$$

$$\begin{aligned}\frac{d\tilde{u}^\theta}{d\tau} = &\left[-\frac{2}{r} + 2\frac{F^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right]\frac{dx^r}{d\tau}\frac{dx^\theta}{d\tau}\\
&+ \left[\cos\theta\sin\theta - \frac{F^2(x_0,p_0)_{,\gamma}}{2C(x_0,p_0,\dot{p}_0)}\sin^2\theta\right]\left(\frac{dx^\phi}{d\tau}\right)^2,\end{aligned} \tag{D.16}$$

$$\frac{d\tilde{u}^\phi}{d\tau} = \left[-\frac{2}{r} + 2\frac{F^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right]\frac{dx^r}{d\tau}\frac{dx^\phi}{d\tau} + \left[\frac{2}{r}\cot\theta + 2\frac{F^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right]\frac{dx^\theta}{d\tau}\frac{dx^\phi}{d\tau}. \tag{D.17}$$

Consequently, the corresponding expansion is given as

$$\tilde{\Theta} = \pm \frac{1}{C(x_0,p_0,\dot{p}_0) r^2 \sqrt{\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) - \frac{r\,F(x_0,p_0)}{C(x_0,p_0,\dot{p}_0)}}} \times \left\{3M + 2r\left[\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right)\left[C(x_0,p_0,\dot{p}_0) + rF^2(x_0,p_0)_{,\gamma}\right] - 1\right]\right\}. \tag{D.18}$$

It is obvious that the convergence of $\tilde{\Theta}$ to Θ increases with decreasing quantization. Furthermore, the proposed quantization has substantial, diverse, and multiple contributions to $\tilde{\Theta}$. The analytical formulation for the evolution of $\tilde{\Theta}$ is given as

$$\begin{aligned} \frac{d\tilde{\Theta}}{d\tau} = {} & \frac{1}{r^3 C^3(x,p,\dot{p})\left[2M + r\left(C(x_0,p_0,\dot{p}_0)\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) - 1\right)\right]} \\ & \times \left\{4r^3(F^2(x_0,p_0)_{,\gamma})^2 \exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right)\right. \\ & \times \left[4M + r\left(C(x_0,p_0,\dot{p}_0)\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) - 2\right)\right] \\ & + 4C(x_0,p_0,\dot{p}_0) r^2 F^2(x_0,p_0)_{,\gamma} \exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) \\ & \times \left[3M + \left(C(x_0,p_0,\dot{p}_0)\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) - 1\right)\right] \\ & + C(x_0,p_0,\dot{p}_0)\left[-9M^2 - 8Mr\left(C(x_0,p_0,\dot{p}_0)\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) - 1\right)\right. \\ & \left.\left. - 2r^2\left(C(x_0,p_0,\dot{p}_0)\exp\left(\frac{4rF^2(x_0,p_0)_{,\gamma}}{C(x_0,p_0,\dot{p}_0)}\right) - 1\right)^2\right]\right\}. \end{aligned} \tag{D.19}$$

In the scenario of vanishing quantization, the evolution equation for the quantum-mechanically revisited expansion, Eq. (D.19), is effectively reduced to that of the expansion which is represented by Eq. (C.13). It is evident that the sign of $d\tilde{\Theta}/d\tau$ is likely positive, particularly due to the apparent positivity of the first two terms. However, it is important to consider that the last term is the only term that may potentially be negative. Assuming that M and $F^2(x_0,p_0)_{,\gamma}$ are constant positive quantities, the negativity of the third term is then conditioned on the positivity of $C(x_0,p_0,\dot{p}_0)$. This analytical analysis allows us to draw the conclusion that the sign of the evolution of the quantum-mechanically revisited expansion would be mainly positive, i.e. the proposed quantization seems to regulate space singularity in GR.

ORCID

A. Tawfik https://orcid.org/0000-0002-1679-0225
Saleh O. Allehabi https://orcid.org/0000-0003-3109-4532
A. A. Alshehri https://orcid.org/0009-0005-8795-2470
D. Mukherjee https://orcid.org/0000-0003-4372-9750
M. Nasar https://orcid.org/0000-0002-3572-1014

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